Stability Analysis of the Formation Control Scheme

Supplement to the M.Sc. Thesis "Validación experimental de controladores multiagentes tipo Quad-Rotor" (J. C. Rodríguez-Cervantes, ITL, 2023)

Julio Cesar Rodríguez-Cervantes

2026

This supplement was written in 2026 and is not part of the thesis defended in August 2023. It completes the stability argument that the thesis only sketched in Theorem 1, which referred to Ríos et al. [IEEE TIE 2019] for the proof. Equation numbers of the form (T-n) refer to the thesis. The proofs follow the methodology of Ríos et al. (2019) and of the other works cited. Where a Lyapunov function or a gain condition comes from a reference, it is cited and not re-derived.

1 Scope and corrections

Checking the thesis against the implemented controllers and the model (T-4) turned up the following errata. The corrected forms are the ones used below.

- (T-8b) should read $\nu_y = u_m(\cos\phi^\star \sin\theta^\star \sin\psi_d - \sin\phi^\star \cos\psi_d)$. This is the second component of $g_\xi(\boldsymbol{\eta})u_m$ and it is consistent with (T-9b).
- (T-8c) should read $\nu_z = u_m \cos\phi^\star \cos\theta^\star - g$. This is the third component of $g_\xi(\boldsymbol{\eta})u_m - \boldsymbol{G}$ and it is consistent with (T-9a).
- (T-17) should read $s = e + k_0 \lceil \varepsilon \rfloor^{3/2}$. With weights $(3, 2, 1)$ for (e, ε, v), the term $\lceil \varepsilon \rfloor^{3/2}$ has weight 3, equal to that of e. Any other exponent makes s non-homogeneous. This is also the form implemented in the firmware.
- (T-14): the implemented STA is $s = \varepsilon + k_0 e$, $\bar\tau = v - k_1 \lceil s \rfloor^{1/2}$, $\dot v = -k_2 \lceil s \rfloor^0$, with no $-k_0\varepsilon$ term in $\bar\tau$. Then $\dot s = \bar\tau + k_0\varepsilon$, so $k_0\varepsilon$ acts as a state-dependent perturbation of the standard STA. This is treated in Proposition 3, and the result is regional, not global.

2 Model, notation and standing assumptions

$\lceil s \rfloor^\gamma := |s|^\gamma \operatorname{sign}(s)$ and $\lceil s \rfloor^0 := \operatorname{sign}(s)$. Classes $\mathcal{K}$, $\mathcal{KL}$ and $\mathcal{KL}_T$ (the last vanishing in finite time) and the properties are used as follows: ISS as in Sontag (1989); US and UES as in Khalil (2002); the finite-time notions UFTS, GUFTS and FT-ISS as in Bhat and Bernstein (2000) and Bernuau et al. (2014). The agent index i is dropped when no confusion arises. Agent n is the leader and $j = 1, \ldots, n-1$ are the followers. The graph is a star centred at the leader. Each agent obeys (T-4):

$$\begin{aligned} \dot{\boldsymbol{\xi}}_1 &= \boldsymbol{\xi}_2, \quad \dot{\boldsymbol{\xi}}_2 = g_\xi(\boldsymbol{\eta}_1)u_m - \boldsymbol{G} - \Lambda_\xi \boldsymbol{\xi}_2 + \boldsymbol{d}_\xi(t), \\ \dot{\boldsymbol{\eta}}_1 &= \boldsymbol{\eta}_2, \quad \dot{\boldsymbol{\eta}}_2 = J\boldsymbol{\tau} + \Xi \boldsymbol{w}_\eta(\boldsymbol{\eta}_2) - \Lambda_\eta \boldsymbol{\eta}_2 + \boldsymbol{d}_\eta(t), \end{aligned} \tag{1}$$

with $\boldsymbol{\xi}_1 = (x, y, z)$, $\boldsymbol{\eta}_1 = (\phi, \theta, \psi)$, $\boldsymbol{G} = (0, 0, g)$, $u_m = u/m$, and

$$g_\xi(\boldsymbol{\eta}_1) = \begin{pmatrix} \cos\phi \sin\theta \cos\psi + \sin\phi \sin\psi \\ \cos\phi \sin\theta \sin\psi - \sin\phi \cos\psi \\ \cos\phi \cos\theta \end{pmatrix}. \tag{2}$$

Let $\hat{\boldsymbol{\xi}}_1, \hat{\boldsymbol{\xi}}_2, \hat{\boldsymbol{\xi}}_3, \hat{\boldsymbol{\eta}}_1, \hat{\boldsymbol{\eta}}_2, \hat{\boldsymbol{\eta}}_3$ be the states of the FT-SMO (T-5). The offsets $C_{n,j}$ are constant. Define $\boldsymbol{\xi}_{dj} := \boldsymbol{\xi}_{1n} + C_{n,j}$ for followers and $\boldsymbol{\xi}_{dn} := \boldsymbol{\xi}_d$ for the leader. Define $\psi_{dj} := \psi_n$ and $\psi_{dn} := \psi_d$. The errors are

$$\boldsymbol{e}_\xi = \boldsymbol{\xi}_1 - \boldsymbol{\xi}_d, \quad \boldsymbol{\varepsilon}_\xi = \hat{\boldsymbol{\xi}}_2 - \dot{\boldsymbol{\xi}}_d, \quad \boldsymbol{e}_\eta = \boldsymbol{\eta}_1 - \boldsymbol{\eta}_d, \quad \boldsymbol{\varepsilon}_\eta = \hat{\boldsymbol{\eta}}_2 - \dot{\boldsymbol{\eta}}_d, \tag{3}$$

with $\boldsymbol{\eta}_d = (\phi^\star, \theta^\star, \psi_d)$. For a follower $\dot{\boldsymbol{\xi}}_d = \boldsymbol{\xi}_{2n}$. The virtual control is $\boldsymbol{w}_\xi := g_\xi(\boldsymbol{\eta}_1) u_m - \boldsymbol{G} - \boldsymbol{\nu}$. The references are

$$\begin{aligned} u_m &= \sqrt{\nu_x^2 + \nu_y^2 + (\nu_z + g)^2}, \\ \phi^\star &= \arcsin\big[u_m^{-1}(\nu_x \sin\psi_d - \nu_y \cos\psi_d)\big], \\ \theta^\star &= \arctan\big[(\nu_z + g)^{-1}(\nu_x \cos\psi_d + \nu_y \sin\psi_d)\big]. \end{aligned} \tag{4}$$

The control (T-10) is

$$\boldsymbol{\nu} = \bar{\boldsymbol{\nu}} + \Lambda_\xi \boldsymbol{\xi}_2 + \ddot{\boldsymbol{\xi}}_d - \hat{\boldsymbol{\xi}}_3, \qquad \boldsymbol{\tau} = J^{-1}\big(\bar{\boldsymbol{\tau}} - \Xi \boldsymbol{w}_\eta + \Lambda_\eta \boldsymbol{\eta}_2 + \ddot{\boldsymbol{\eta}}_d - \hat{\boldsymbol{\eta}}_3\big), \tag{5}$$

where $\bar{\nu}_q = k_{0q} \int e_q + k_{1q} e_q + k_{2q} \varepsilon_q$ for $q = x, y, z$ (T-11), and each companion matrix

$$A_q = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ k_{0q} & k_{1q} & k_{2q} \end{pmatrix} \tag{6}$$

is Hurwitz (T-12). Since the characteristic polynomial of A_q is $s^3 - k_{2q} s^2 - k_{1q} s - k_{0q}$, by the Routh–Hurwitz criterion A_q is Hurwitz if and only if $k_{0q}, k_{1q}, k_{2q} < 0$ and $k_{1q} k_{2q} > -k_{0q}$, with the convention $e =$ measurement $-$ reference used here; with the opposite convention $e =$ reference $-$ measurement, as in the flight firmware, the same gains appear with positive sign. The same holds for the attitude PID (T-13). The attitude controllers $\bar{\tau}_l$, $l = \phi, \theta, \psi$, are (a) PID (T-13), (b) STA (T-14, as implemented), (c) continuous twisting (T-15), (d) STSMC (T-16) and (e) NTSMC (T-17, corrected).

Assumption A1. *Thesis Assumptions 3–4: the desired trajectories are C^∞ and bounded together with their derivatives up to order four (needed for $\ddot{\boldsymbol{\xi}}_d$ and, through (4), for $\ddot{\boldsymbol{\eta}}_d$), and the disturbances $\boldsymbol{d}_\xi, \boldsymbol{d}_\eta$ are uniformly bounded and Lipschitz with known bounds.*

Assumption A2. *$\nu_z + g > 0$ along solutions (for instance by saturating $\bar{\nu}_z$), so $\theta^\star$ is well defined and $u_m > 0$. The angle $\phi^\star$ is always well defined, because by Cauchy–Schwarz $(\nu_x \sin\psi_d - \nu_y \cos\psi_d)^2 \le (\nu_x^2 + \nu_y^2)(\sin^2\psi_d + \cos^2\psi_d) \le u_m^2$.*

Assumption A3. *The thrust is bounded: $u_{mi} \le \bar{u}$.*

Assumption A4. *The feed-forward signals $\ddot{\boldsymbol{\xi}}_{di}$ (for followers, the leader acceleration, transmitted or estimated) and $\ddot{\boldsymbol{\eta}}_{di}$ are available and bounded.*

Assumption A5. *Solutions of the closed loop, including the observer and controller states, exist and remain bounded on $[0, T_0]$ (no finite escape).*

Plausibility of A5 (not a proof). On $[0, T_0]$ the cancellation terms in (5) that involve $\boldsymbol{\xi}_2, \boldsymbol{\eta}_2$ use the measured states, so the residual forcing consists of disturbances and estimation errors. These are bounded by A1 and Lemma 2. The position loop is $\ddot{\boldsymbol{\xi}} = g_\xi u_m - \boldsymbol{G} - \Lambda_\xi \dot{\boldsymbol{\xi}} + \boldsymbol{d}_\xi$, a linear system driven by a bounded input (A3), so it cannot escape in finite time. In every attitude controller $\bar{\tau}$ has at most linear growth in (e, ε, v), because the fractional powers are sublinear and $\dot{v}$ is linear or bounded. A system with linearly bounded right-hand side and bounded forcing has no finite escape. This is the standard argument of Ríos et al. (2019). It is kept as an assumption because (T-5) is not reproduced here.

Lemma 1 (reference attitude)**.** *Under A2,* (4) *gives* $\phi^\star \in [-\pi/2, \pi/2]$, $\theta^\star \in (-\pi/2, \pi/2)$, *and* $g_\xi(\boldsymbol{\eta}_d) u_m - \boldsymbol{G} = \boldsymbol{\nu}$ *with* $\boldsymbol{\eta}_d = (\phi^\star, \theta^\star, \psi_d)$.

Proof. Put $a = \nu_x \cos\psi_d + \nu_y \sin\psi_d$, $b = \nu_z + g > 0$ and $c = \nu_x \sin\psi_d - \nu_y \cos\psi_d$, so $u_m^2 = a^2 + b^2 + c^2$. Then $\sin\phi^\star = c/u_m$ and, since $\cos\phi^\star \geq 0$, $\cos\phi^\star = \sqrt{a^2 + b^2}/u_m$. Also $\tan\theta^\star = a/b$ with $\cos\theta^\star > 0$ gives $\cos\theta^\star = b/\sqrt{a^2 + b^2}$ and $\sin\theta^\star = a/\sqrt{a^2 + b^2}$. Hence $u_m \cos\phi^\star \cos\theta^\star = b$ and $u_m \cos\phi^\star \sin\theta^\star = a$. From (2), the first component of $g_\xi(\boldsymbol{\eta}_d) u_m$ is

$$u_m \cos\phi^\star \sin\theta^\star \cos\psi_d + u_m \sin\phi^\star \sin\psi_d = a \cos\psi_d + c \sin\psi_d = \nu_x,$$

and the second is $a \sin\psi_d - c \cos\psi_d = \nu_y$. The third equals $b = \nu_z + g$. □

3 Observer and closed loop for $t \geq T_0$

Lemma 2 (FT-SMO, cited)**.** *Under A1 there exist gains of the third-order observer (T-5), with injections* $\lceil\cdot\rfloor^{2/3}, \lceil\cdot\rfloor^{1/3}, \lceil\cdot\rfloor^{0}$ *chosen from the Lipschitz bounds of* $\boldsymbol{d}_\xi, \boldsymbol{d}_\eta$, *such that the estimation error is UFTS. Hence there is a finite* T_0, *bounded for bounded initial estimation errors, with* $\hat{\boldsymbol{\xi}}_1 = \boldsymbol{\xi}_1$, $\hat{\boldsymbol{\xi}}_2 = \boldsymbol{\xi}_2$, $\hat{\boldsymbol{\xi}}_3 = \boldsymbol{d}_\xi$, $\hat{\boldsymbol{\eta}}_1 = \boldsymbol{\eta}_1$, $\hat{\boldsymbol{\eta}}_2 = \boldsymbol{\eta}_2$, $\hat{\boldsymbol{\eta}}_3 = \boldsymbol{d}_\eta$ *for* $t \geq T_0$ *(T-6).*

The observer equations and gains are those of (T-5); they coincide with the FT-SMO of Ríos et al. (2017) and Ríos et al. (2019, Theorem 1), where this result is proved (see also Levant, 2003). It is not repeated here. The error dynamics are driven only by the derivative of the disturbance, which is bounded by A1.

For $t \geq T_0$ we have $\boldsymbol{\varepsilon}_\xi = \boldsymbol{\xi}_2 - \dot{\boldsymbol{\xi}}_d$ and, by the definition of $\boldsymbol{w}_\xi$, $g_\xi u_m - \boldsymbol{G} = \boldsymbol{\nu} + \boldsymbol{w}_\xi$. Substituting (5) into the error dynamics (T-7) gives

$$\dot{\boldsymbol{e}}_\xi = \boldsymbol{\varepsilon}_\xi, \quad \dot{\boldsymbol{\varepsilon}}_\xi = \bar{\boldsymbol{\nu}} + \boldsymbol{w}_\xi, \qquad \dot{\boldsymbol{e}}_\eta = \boldsymbol{\varepsilon}_\eta, \quad \dot{\boldsymbol{\varepsilon}}_\eta = \bar{\boldsymbol{\tau}}. \tag{7}$$

Indeed $\dot{\boldsymbol{\varepsilon}}_\xi = \boldsymbol{\nu} + \boldsymbol{w}_\xi - \Lambda_\xi \boldsymbol{\xi}_2 + \boldsymbol{d}_\xi - \ddot{\boldsymbol{\xi}}_d = \bar{\boldsymbol{\nu}} + \boldsymbol{w}_\xi + (\boldsymbol{d}_\xi - \hat{\boldsymbol{\xi}}_3)$, and $\dot{\boldsymbol{\varepsilon}}_\eta = \bar{\boldsymbol{\tau}} + (\boldsymbol{d}_\eta - \hat{\boldsymbol{\eta}}_3)$ after the terms $\Xi \boldsymbol{w}_\eta$, $\Lambda_\eta \boldsymbol{\eta}_2$, $\ddot{\boldsymbol{\eta}}_d$ cancel. The last brackets vanish for $t \geq T_0$. Note that the $(\boldsymbol{e}_\eta, \boldsymbol{\varepsilon}_\eta)$ subsystem does not depend on the position states. This relies on the exact use of $\ddot{\boldsymbol{\eta}}_d$ in (5) (A4), and it is what makes the scheme a cascade. Note that $\boldsymbol{\eta}_d$ itself depends on the position states through $\boldsymbol{\nu}$; what is independent of them is the error dynamics of $(\boldsymbol{e}_\eta, \boldsymbol{\varepsilon}_\eta)$, provided that $\ddot{\boldsymbol{\eta}}_d$ is available (A4).

Lemma 3 (coupling bound)**.** *Under A2–A3,* $\boldsymbol{w}_\xi = \big(g_\xi(\boldsymbol{\eta}_1) - g_\xi(\boldsymbol{\eta}_d)\big) u_m$ *and* $\|\boldsymbol{w}_\xi\| \leq L_g \bar{u} \, \|\boldsymbol{e}_\eta\|$ *with* $L_g = 6$.

Proof. By Lemma 1, $\boldsymbol{\nu} = g_\xi(\boldsymbol{\eta}_d) u_m - \boldsymbol{G}$, which gives the identity. Every partial derivative of a component of g_ξ is a sum of at most two products of sines and cosines (for example $\partial_\phi g_{\xi,1} = -\sin\phi \sin\theta \cos\psi + \cos\phi \sin\psi$ and $\partial_\theta g_{\xi,1} = \cos\phi \cos\theta \cos\psi$), so its magnitude is at most 2. Hence $\|\partial g_\xi / \partial \boldsymbol{\eta}\|_2 \leq \|\partial g_\xi / \partial \boldsymbol{\eta}\|_F \leq \sqrt{9 \cdot 4} = 6$. Since $\mathbb{R}^3$ is convex, $g_\xi(\boldsymbol{\eta}_1) - g_\xi(\boldsymbol{\eta}_d) = \int_0^1 \frac{\partial g_\xi}{\partial \boldsymbol{\eta}}(\boldsymbol{\eta}_d + s\boldsymbol{e}_\eta)\, ds \; \boldsymbol{e}_\eta$, and $u_m \leq \bar{u}$ completes the proof. The constant 6 is conservative. □

4 Position loop

Proposition 1 (ISS w.r.t. $\boldsymbol{e}_\eta$)**.** *Let $t_0 \geq T_0$ and $q \in \{x, y, z\}$. With $\chi_q = (\int e_q,\, e_q,\, \varepsilon_q)^\top$ and $B = (0, 0, 1)^\top$, the closed loop (7) gives $\dot{\chi}_q = A_q \chi_q + B\, w_q$, where w_q is the q-th component of $\boldsymbol{w}_\xi$. There exist $\kappa \geq 1$ and $\lambda > 0$, depending only on A_q, such that for $t \geq t_0$*

$$\|\chi_q(t)\| \leq \kappa e^{-\lambda(t-t_0)} \|\chi_q(t_0)\| + \frac{\kappa}{\lambda} L_g \bar{u}\, \|\boldsymbol{e}_\eta\|_\infty, \tag{8}$$

with $\|\boldsymbol{e}_\eta\|_\infty = \sup_{[t_0,t]} \|\boldsymbol{e}_\eta\|$. If $\boldsymbol{e}_\eta \equiv 0$ on $[t_1, \infty)$, then $\|\chi_q(t)\| \leq \kappa e^{-\lambda(t-t_1)} \|\chi_q(t_1)\|$, i.e. the position loop is exponentially stable once its input $\boldsymbol{e}_\eta$ vanishes.

Proof. From $\frac{d}{dt}\int e_q = e_q$, $\dot{e}_q = \varepsilon_q$ and $\dot{\varepsilon}_q = \bar{\nu}_q + w_q = k_{0q} \int e_q + k_{1q} e_q + k_{2q} \varepsilon_q + w_q$ we get $\dot{\chi}_q = A_q \chi_q + B w_q$. Since A_q is Hurwitz, $\|e^{A_q t}\| \leq \kappa e^{-\lambda t}$ for some $\kappa \geq 1$, $\lambda > 0$ (Khalil, 2002). Variation of constants gives

$$\chi_q(t) = e^{A_q(t-t_0)} \chi_q(t_0) + \int_{t_0}^{t} e^{A_q(t-s)} B w_q(s)\, ds.$$

Since $\|B\| = 1$, $|w_q| \leq \|\boldsymbol{w}_\xi\| \leq L_g \bar{u} \|\boldsymbol{e}_\eta\|$ by Lemma 3, and $\int_{t_0}^{t} e^{-\lambda(t-s)} ds \leq 1/\lambda$, estimate (8) follows. If $\boldsymbol{e}_\eta \equiv 0$ on $[t_1, \infty)$, then $w_q \equiv 0$ there and the first term of the variation-of-constants formula alone remains. □

5 Attitude loop

For a vector field F and weights $r = (r_1, \ldots, r_m) > 0$, homogeneity of degree q means

$$F_k(\lambda^{r_1} x_1, \ldots, \lambda^{r_m} x_m) = \lambda^{r_k + q} F_k(x) \quad \text{for all } \lambda > 0.$$

Discontinuous systems are understood in the sense of Filippov, with $\lceil s \rfloor^0$ replaced by the set-valued sign. Two facts from Bernuau et al. (2014) are used. (H1) A homogeneous system of negative degree that is globally asymptotically stable is GUFTS. (H2) For $\dot{x} = F(x, w)$ homogeneous of degree q with weights r for x and $\tilde{r}$ for w, global asymptotic stability at $w = 0$ and $\min \tilde{r} > 0$ imply ISS, and if moreover $q < 0$ the system is FT-ISS.

Write e, ε, τ for one channel l, and v for the state of the controller. For $t \geq T_0$ the channel satisfies $\dot{e} = \varepsilon$, $\dot{\varepsilon} = \bar{\tau}$ by (7).

Proposition 2 (attitude loop, controllers a, c, d, e)**.** *Let $t_0 \geq T_0$ and assume the gains satisfy the conditions of the cited references.*

- *a) PID: with $\chi = (\int e, e, \varepsilon)$, $\dot{\chi} = A\chi$ and $\|\chi(t)\| \leq \kappa e^{-\lambda(t-t_0)} \|\chi(t_0)\|$ (GUES).*
- *c) TC: $x = (e, \varepsilon, v) = 0$ is GUFTS, with settling time $T \leq 5V(x_0)^{1/5}/\kappa_V$.*
- *d) STSMC: $x = 0$ is GUFTS, with settling time $T \leq 4V(x_0)^{1/4}/\kappa_V$.*
- *e) NTSMC: $x = 0$ is GUFTS.*
- *f) For c–e, with $\dot{\varepsilon} = \bar{\tau} + \rho(t)$ and ρ measurable and essentially bounded, the system is FT-ISS w.r.t. ρ.*

Proof. *a)* $\dot\chi = A\chi$ with A Hurwitz (T-13), so $\|\chi(t)\| = \|e^{A(t-t_0)}\chi(t_0)\| \le \kappa e^{-\lambda(t-t_0)}\|\chi(t_0)\|$ for all $\chi(t_0)$.

c) The closed loop is

$$\dot e = \varepsilon, \quad \dot\varepsilon = v - k_0\lceil e\rfloor^{1/3} - k_1\lceil\varepsilon\rfloor^{1/2}, \quad \dot v \in -k_2\,\mathrm{sign}(e) - k_3\,\mathrm{sign}(\varepsilon).$$

Take weights $(3,2,1)$ for (e,ε,v) and degree $q=-1$. Then $\dot e = \varepsilon$ scales as $\lambda^2 = \lambda^{3+q}$, and in $\dot\varepsilon$ the terms v, $\lceil e\rfloor^{1/3}$, $\lceil\varepsilon\rfloor^{1/2}$ scale as $\lambda^1 = \lambda^{2+q}$. The sign terms in $\dot v$ scale as $\lambda^0 = \lambda^{1+q}$. So the system is homogeneous of degree -1. For gains satisfying the conditions of Torres-González et al. (2017), see also Ríos et al. (2019), a positive definite, radially unbounded Lyapunov function

$$V = \beta_1|e|^{5/3} + \beta_2 e\varepsilon + \beta_3|\varepsilon|^{5/2} + \beta_4 e\lceil v\rfloor^2 - \beta_5\varepsilon v^3 + \beta_6|v|^5, \qquad \beta_i > 0,$$

homogeneous of degree 5, is constructed there with $\dot V \le -\kappa_V V^{4/5}$ along solutions. This gives GUFTS. Moreover $\frac{d}{dt}V^{1/5} = \frac{1}{5}V^{-4/5}\dot V \le -\kappa_V/5$ while $V>0$, so V vanishes no later than $T \le 5V(x_0)^{1/5}/\kappa_V$.

d) In the coordinates $x=(e,\varepsilon,v)$ with $s = \varepsilon + k_0\lceil e\rfloor^{2/3}$ the closed loop is

$$\dot e = \varepsilon, \quad \dot\varepsilon = v - k_1\lceil s\rfloor^{1/2}, \quad \dot v \in -k_2\,\mathrm{sign}(s).$$

The variable s has weight 2, since ε has weight 2 and $\lceil e\rfloor^{2/3}$ has weight $3\cdot\frac{2}{3}=2$. Hence $\lceil s\rfloor^{1/2}$ has weight $1 = 2+q$ and the sign term has weight $0 = 1+q$. The system is homogeneous of degree -1 with weights $(3,2,1)$. The closed loop in (e,ε,v) is discontinuous, because of the term $\mathrm{sign}(s)$, and is understood in the sense of Filippov; it contains no singular terms, since only $\lceil e\rfloor^{2/3}$ appears and not its derivative. The Lyapunov function below is continuous but not differentiable on $\{e=0\}$; its decrease is established in the cited works with non-smooth analysis. For gains satisfying the conditions of Fridman et al. (2015), see also Ríos et al. (2019), the quadratic form $V = \zeta^\top P\zeta$ with $P = P^\top > 0$ and $\zeta = (\lceil e\rfloor^{2/3}, s, \lceil v\rfloor^2)$, whose entries all have weight 2, so V has degree 4, satisfies $\dot V \le -\kappa_V V^{3/4}$. As in c), this gives GUFTS and $T \le 4V(x_0)^{1/4}/\kappa_V$.

e) With the corrected (T-17), $s = e + k_0\lceil\varepsilon\rfloor^{3/2}$, the closed loop is

$$\dot e = \varepsilon, \quad \dot\varepsilon = v - k_1\lceil s\rfloor^{1/3}, \quad \dot v \in -k_2\,\mathrm{sign}(s).$$

For weights $(3,2,1)$ the term $\lceil\varepsilon\rfloor^{3/2}$ has weight $2\cdot\frac{3}{2}=3$, equal to the weight of e, so s is homogeneous of weight 3. Then $\lceil s\rfloor^{1/3}$ has weight $1 = 2+q$, $\dot e = \varepsilon$ has weight $2 = 3+q$, and the sign term has weight $0 = 1+q$. The system is homogeneous of degree -1. This is exactly the continuous terminal sliding-mode controller of Kamal et al. (2016), $u = -k_1L^{2/3}\lceil\phi_L\rfloor^{1/3} + z$, $\dot z = -k_2L\lceil\phi_L\rfloor^0$ with $\phi_L = x_1 + \alpha L^{-1/2}\lceil x_2\rfloor^{3/2}$. For gains satisfying their conditions, Kamal et al. (2016, Theorem 1) construct a continuously differentiable, homogeneous and strict Lyapunov function, which proves GUFTS. The function is not reproduced here.

f) The perturbation ρ enters the ε-equation, whose weighted scaling is $\lambda^{2+q} = \lambda^1$. Assigning ρ the weight $\tilde r = 1$, the perturbed system $F(x,\rho)$ is homogeneous of degree -1 in the sense of (H2). Since $\min\tilde r = 1 > 0$, $q<0$, and the unperturbed system is GAS by c)–e), (H2) gives FT-ISS: there exist $\beta \in \mathcal{KL}_T$ and $\gamma\in\mathcal{K}$ with $\|x(t)\| \le \beta(\|x(t_0)\|, t-t_0) + \gamma(\|\rho\|_\infty)$. □

Remark 1. In c)–f), a bounded compensation residual, for example $\rho = \boldsymbol{d}_\eta - \hat{\boldsymbol{\eta}}_3$ on $[0,T_0]$ (bounded on $[0,T_0]$ by A5), leaves the attitude state bounded by $\gamma(\|\rho\|_\infty)$ plus a decaying term. Perturbations that enter through the v-channel have weight 0 and are not covered by (H2). They are handled in the cited works through gain conditions for Lipschitz perturbations, which are not restated here.

Proposition 3 (STA as implemented)**.** *Take controller b): $s = \varepsilon + k_0 e$, $\bar\tau = v - k_1\lceil s\rfloor^{1/2}$, $\dot v = -k_2\lceil s\rfloor^0$, with $k_0, k_1, k_2 > 0$. Let $\bar v := v + k_0\varepsilon$. For $t \geq T_0$,*

$$\dot s = -k_1\lceil s\rfloor^{1/2} + \bar v, \qquad \dot{\bar v} = -k_2\lceil s\rfloor^0 + k_0\bar\tau.$$

Let $K \subset \mathbb{R}^3$ be compact and let $\varrho_K := k_0 \max_{(e,\varepsilon,v)\in K} |v - k_1\lceil \varepsilon + k_0 e\rfloor^{1/2}|$. Suppose k_1, k_2 satisfy the gain conditions of the super-twisting algorithm for a Lipschitz perturbation with constant ϱ_K, for instance $k_1 = 1.5\sqrt{\varrho_K}$ and $k_2 = 1.1\varrho_K$ (Seeber and Horn, 2017), which is the setting of Table 1 of the thesis with $\zeta = \varrho_K$. Then every solution that stays in K for $t \geq T_0$ (this is assumed: a positively invariant set K is not constructed here) reaches $s = \bar v = 0$ in finite time T_s, and afterwards $(e, \varepsilon, v) \to 0$ exponentially.

Proof. From $\dot e = \varepsilon$ and $\dot\varepsilon = \bar\tau$ we get $\dot s = \bar\tau + k_0\varepsilon = -k_1\lceil s\rfloor^{1/2} + v + k_0\varepsilon$, and $\dot{\bar v} = \dot v + k_0\dot\varepsilon = -k_2\lceil s\rfloor^0 + k_0\bar\tau$. Set $\delta(t) := k_0\bar\tau(t)$ along the solution. Since the solution stays in K, $|\delta(t)| \leq \varrho_K$. The pair $(s, \bar v)$ is therefore a super-twisting algorithm whose second equation carries a measurable perturbation bounded by ϱ_K (equivalently, the perturbation $\rho = k_0\varepsilon$ in the first equation, whose derivative $\dot\rho = \delta$ is bounded by ϱ_K). The cited gain conditions are sufficient for finite-time convergence of $(s, \bar v)$ to 0 for every such perturbation. This gives $T_s < \infty$ with $s = \bar v = 0$ for $t \geq T_s$. There, $v = -k_0\varepsilon$ and $\bar\tau = v - k_1\lceil 0\rfloor^{1/2} = -k_0\varepsilon$, so $\dot\varepsilon = -k_0\varepsilon$. Also $s = 0$ gives $\varepsilon = -k_0 e$ and hence $\dot e = -k_0 e$. Therefore e, ε, v decay as $e^{-k_0(t-T_s)}$. The sliding dynamics are UES, and $|k_0\bar\tau| = k_0^2|\varepsilon|$ decreases, so the bound by ϱ_K remains valid. □

Remark 2. Proposition 3 is a regional (semi-global) result, and it is conditional on the solution staying in K. The gains must be raised with the size of K through ϱ_K. Adding the missing term, $\bar\tau = v - k_1\lceil s\rfloor^{1/2} - k_0\varepsilon$, gives $\dot s = -k_1\lceil s\rfloor^{1/2} + v$ and $\dot v = -k_2\lceil s\rfloor^0$. This is the standard unperturbed STA, which is globally finite-time stable for $k_1, k_2 > 0$ (Moreno and Osorio, 2012). Then $s = 0$ gives $\dot e = -k_0 e$, and the result becomes global. The thesis (T-14) as implemented lacks this term.

6 Main result

Theorem 1 (restated and corrected)**.** *Under A1–A5, with the controls (T-9)–(T-11),* (4), (5) *and any attitude controller a)–e), with gains satisfying the conditions of Propositions 1–3 and of the references cited there (for b) also under the hypothesis of Proposition 3 that solutions remain in K), the errors $(\boldsymbol{e}_\xi, \boldsymbol{\varepsilon}_\xi, \boldsymbol{e}_\eta, \boldsymbol{\varepsilon}_\eta)$ of every agent converge to zero. The position errors converge exponentially. The attitude errors converge in finite time for c–e and exponentially for a, b. Consequently the followers reach the pattern $\boldsymbol{\xi}_{1j} - \boldsymbol{\xi}_{1n} = C_{n,j}$ and the leader tracks $\boldsymbol{\xi}_d$.*

Proof. (i) Exact compensation. By Lemma 2, for $t \geq T_0$ the loop is (7). By A5 the state at T_0 is finite. The attitude subsystem $(\boldsymbol{e}_\eta, \boldsymbol{\varepsilon}_\eta)$ does not depend on the position states (Section 3).

(ii) Cascade. For c–e, Proposition 2 gives a finite $T_1 \geq T_0$, depending on the state at T_0, with $\boldsymbol{e}_\eta = \boldsymbol{\varepsilon}_\eta \equiv 0$ for $t \geq T_1$. By Lemma 3, $\boldsymbol{w}_\xi \equiv 0$ on $[T_1, \infty)$. By (8), $\chi_q(T_1)$ is bounded, so $\|\chi_q(t)\| \leq \kappa e^{-\lambda(t-T_1)}\|\chi_q(T_1)\|$ (UES). For a, b, there are $c, \mu > 0$ with $\|\boldsymbol{e}_\eta(t)\| \leq ce^{-\mu(t-T_0)}$ (Proposition 2a), and for b Proposition 3, where the finite-time phase before T_s only adds a bounded transient). Inserting this in the variation-of-constants formula of Proposition 1,

$$\|\chi_q(t)\| \leq \kappa e^{-\lambda(t-T_0)}\|\chi_q(T_0)\| + \kappa L_g\bar u\, c\int_{T_0}^{t} e^{-\lambda(t-s)}e^{-\mu(s-T_0)}ds,$$

and $\int_0^\tau e^{-\lambda(\tau-s)}e^{-\mu s}ds \leq \tau e^{-\min(\lambda,\mu)\tau} \leq (e\delta)^{-1}e^{-(\min(\lambda,\mu)-\delta)\tau}$ for any $\delta \in (0, \min(\lambda, \mu))$. Hence $\chi_q \to 0$ exponentially. The same conclusion follows from the cascade lemma for ISS systems (Khalil, 2002, Lemma 4.7). So $\boldsymbol{e}_\xi, \boldsymbol{\varepsilon}_\xi \to 0$ for each agent, and $\boldsymbol{\varepsilon}_\eta \to 0$ is part of the attitude result.

(iii) Formation. For a follower j, $\boldsymbol{e}_{\xi j} = \boldsymbol{\xi}_{1j} - \boldsymbol{\xi}_{1n} - C_{n,j} \to 0$ gives the geometric pattern, and for the leader $\boldsymbol{e}_{\xi n} \to 0$ gives $\boldsymbol{\xi}_{1n} \to \boldsymbol{\xi}_d$. In the star graph each follower depends only on the leader, through the bounded signals $\boldsymbol{\xi}_{1n}, \boldsymbol{\xi}_{2n}, \ddot{\boldsymbol{\xi}}_n, \psi_n$ (A4), and the leader does not depend on any follower. With the exact feed-forward (5) the error system of each agent is the one analysed in Propositions 1–3, which does not contain the states of other agents. There are therefore no interconnection loops, and the convergence of the whole formation follows agent by agent. □

7 What is not proven

The following are outside this analysis: the discrete-time implementation, communication delays and packet loss (Crazyradio), actuator saturation beyond A3 (including the effect of saturating $\bar{\nu}_z$ on the linear position analysis), and the approximation of $\ddot{\boldsymbol{\eta}}_d$. For the last, only ISS-type ultimate bounds follow from Proposition 2f) when the residual is bounded. A4 assumes the boundedness of the signals $\ddot{\boldsymbol{\eta}}_d$ along the closed loop, and this is not proven. The gain conditions for c–e are taken from the cited works and are not checked for the gains used in the experiments.

References

[1] H. Ríos, R. Falcón, O. A. González, and A. Dzul, "Continuous sliding-mode control strategies for quadrotor robust tracking: Real-time application," *IEEE Trans. Ind. Electron.*, vol. 66, no. 2, pp. 1264–1272, 2019.

[2] H. Ríos, J. González-Sierra, and A. Dzul, "Robust tracking output-control for a quad-rotor: A continuous sliding-mode approach," *J. Franklin Inst.*, vol. 354, no. 15, pp. 6672–6691, 2017.

[3] A. Levant, "Higher-order sliding modes, differentiation and output-feedback control," *Int. J. Control*, vol. 76, no. 9–10, pp. 924–941, 2003.

[4] J. A. Moreno and M. Osorio, "Strict Lyapunov functions for the super-twisting algorithm," *IEEE Trans. Autom. Control*, vol. 57, no. 4, pp. 1035–1040, 2012.

[5] R. Seeber and M. Horn, "Stability proof for a well-established super-twisting parameter setting," *Automatica*, vol. 84, pp. 241–243, 2017.

[6] V. Torres-González, T. Sánchez, L. Fridman, and J. A. Moreno, "Design of continuous twisting algorithm," *Automatica*, vol. 80, pp. 119–126, 2017.

[7] L. Fridman, J. A. Moreno, B. Bandyopadhyay, S. Kamal, and A. Chalanga, "Continuous nested algorithms: The fifth generation of sliding mode controllers," in *Recent Advances in Sliding Modes: From Control to Intelligent Mechatronics*, X. Yu and M. Ö. Efe, Eds. Cham: Springer, 2015, ch. 2, pp. 5–35.

[8] S. Kamal, J. A. Moreno, A. Chalanga, B. Bandyopadhyay, and L. Fridman, "Continuous terminal sliding-mode controller," *Automatica*, vol. 69, pp. 308–314, 2016.

[9] E. D. Sontag, "Smooth stabilization implies coprime factorization," *IEEE Trans. Autom. Control*, vol. 34, no. 4, pp. 435–443, 1989.

[10] E. Bernuau, D. Efimov, W. Perruquetti, and A. Polyakov, "On homogeneity and its application in sliding mode control," *J. Franklin Inst.*, vol. 351, no. 4, pp. 1866–1901, 2014.

[11] S. P. Bhat and D. S. Bernstein, "Finite-time stability of continuous autonomous systems," *SIAM J. Control Optim.*, vol. 38, no. 3, pp. 751–766, 2000.

[12] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Prentice Hall, 2002.

[13] J. C. Rodríguez-Cervantes, "Validación experimental de controladores multiagentes tipo Quad-Rotor," M.Sc. thesis, Instituto Tecnológico de La Laguna (ITL), 2023.